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Spin-Spin Correlation Function in the Two-Dimensional Ising Model with Linear Defects. II. $T > T_c$

Lee-Fen Ko[1]

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The dispersion expansion for the spin correlation function in the two-dimensional Ising model with linear defects above T_c is derived. The asymptotic behavior is computed by a steepest descent analysis. The lattice is divided into four domains with different asymptotic behaviors. In particular, the correlation length inside certain domains is a function of the defect.

KEY WORDS: 2D spin system; Ising model; linear defect; spin correlation.

1. INTRODUCTION

Since the famous solution of Onsager[1] demonstrating exactly that there is a phase transition in the two-dimensional Ising model, it has been natural to divide the discussion of such systems into two parts: the low-temperature and the high-temperature phases. There is also the case at the critical temperature, which requires different techniques and will not be considered here.

It was shown by Kramers and Wannier[2] that the high- and low-temperature phases are related by a dual transformation, namely, a lattice with horizontal and vertical bonds E_1/kT and E_2/kT are transformed into the dual lattice with bonds $(E_2/kT)^*$ and $(E_1/kT)^*$, satisfying

$$\sinh\frac{2E_i}{kT}\sinh\left(\frac{2E_i}{kT}\right)^* = 1, \qquad i = 1, 2 \tag{1.1}$$

[1] Institute for Theoretical Physics, State University of New York, Stony Brook, New York 11794, and Department of Physics, Clarkson University, Potsdam, New York 13676.

The critical temperature of the Onsager Ising lattice can be obtained by requiring that the critical point be the self-dual point,

$$\sinh\frac{2E_1}{kT_c}\sinh\frac{2E_2}{kT_c}=1 \tag{1.2}$$

In part I of this work[3] we studied an Ising model that is not self-dual; nevertheless, it has an order–disorder phase transition at the same critical temperature as in Onsager's lattice given above. The model studied is a defect model, in which the nearest neighbor interaction in the center rows on the two-dimensional lattice is modified, as shown in Fig. 1 in paper I; the boundary conditions are periodic for the pure-phase case.[4] This model contains two special cases[5] which are dual to one another, the line-defect model ($E_2'\to\infty$) and the ladder-defect model ($E_1'=E_1$).

In I, we computed the spin–spin correlation function in the low-temperature regime. In this paper, we study the spin–spin correlation in the high-temperature regime.

The dispersion expansion for the spin–spin correlation function below T_c was given in I; it splits into the case where the spins are on the same side of the defect and the case where they are on opposite sides. It was found that in the scaling limit, the defect couplings E_1' and E_2' appear in the correlation only in the defect parameter τ,

$$\tau=\operatorname{sgn}(T-T_c)\tanh[2(2E_1'-E_1-E_2'^*)/kT_c] \tag{1.3}$$

The contours of τ in E_1'–E_2' space are shown in Fig. 2 in paper I for $E_1=E_2\cong 0.44$ in units of kT_c. The function τ is antisymmetric if it is reflected with respect to the point $E_1'=E_1/2$, $E_2'=0$, which is marked with an open circle in the figure. The general tendency of τ, which takes values between -1 and 1, is that it rises sharply near zero for increasing E_1' and its magnitude drops sharply near 1 for increasing $|E_2'|$. At the line $E_2'=0$, which is the half-plane case, τ is discontinuous, depending on whether the limit is approached ferromagnetically or antiferromagnetically. The pure system is at the solid circle, which is embedded in a smooth curve; therefore an interesting phenomenon is that for any system with nonzero value of E_2', say, we can find an E_1' to cancel τ, so that it is equivalent to the pure system.

We also computed the asymptotic decay in I for large separations between the spins and from the defect. It was found that if $\tau>0$ the correlation length in certain domains becomes defect dependent; fix one spin $\sigma_{y_1,0}$; then the domain for the second spin $\sigma_{y_2,x}$ where this occurs is defined by the defect, a straight line, and a parabola given by

$$y_2 + \tau\bar{\tau}^{-1}x = y_1 \tag{1.4}$$

$$(\tau x - \bar{\tau}y_2 + \bar{\tau}y_1)^2 = 4\tau y_1(\bar{\tau}x + \tau y_2) \tag{1.5}$$

where $\bar{\tau}^2 = 1 - \tau^2$. This is shown in Fig. 3 in paper I.

These features are shared by the high-temperature case. From the sign factor in (1.3), it is clear that the effects of defects in the two phases are opposite. The explicit results in this paper are given in (3.5) and (4.2) for the dispersion expansion on the lattice and in the scaling limit, and in (4.4)–(4.10) for the asymptotic behavior. We also give the dispersion expansion for the case for one spin on the defect for the line-defect model in Appendix C.

2. FORMULATION OF THE PROBLEM

In ref. 6 it was shown, using the transfer matrix method of Onsager and Kaufman,[7] that the spin–spin correlation function is expressed in terms of a block Toeplitz matrix

$$\langle \sigma_{l,0}\sigma_{m,n}\rangle = \det{}^{1/2}\begin{pmatrix} A_0 + A_{11} & A_{12} \\ A_{21} & A_0 + A_{22} \end{pmatrix} \tag{2.1}$$

in which the submatrices are 2×2 block Toeplitz matrices given below:

$$(A_0)_{kj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-i(k-j)\phi}\begin{pmatrix} 0 & C(\phi) \\ -C(\phi)^{-1} & 0 \end{pmatrix} \tag{2.2}$$

where the function $C(\phi)$ is defined in (I.A.1); and

$$(A_{pq})_{kj} = \sum_{s=1}^{2} (A^s_{pq})_{kj} \tag{2.3a}$$

$$(A^s_{pq})_{kj} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-i(k-j)\phi} X^s_{pq}(\phi)\, b^s_{pq}(\phi) \tag{2.3b}$$

with $X^s_{pq}(\phi)$ a function and $b^s_{pq}(\phi)$ a 2×2 matrix given in (I.A.4) and (I.2.7), respectively; and p, $q = 1, 2$ and k, $j = 0,\ldots, \infty$.

In ref. 3, we calculated (2.1) by expanding the determinant around A_0, which involves explicit use of $\det(A_0)$ and A_0^{-1}. That procedure was practical for $T < T_c$ because the function $C(\phi)$ satisfies the condition necessary to evaluate $\det(A_0)$ and A_0^{-1} from Szegö's theorem and Wiener–Hopf method,[8] namely, $\ln C(\phi)$ is continuous and periodic. For $T > T_c$, however, $C(\phi)$ does not satisfy the condition; it is $\tilde{C}(\phi) \equiv e^{i\phi}C(\phi)$ that does. A known method[8] of attack in this case is to devise a comparison matrix containing $\bar{C}(\phi)$. This comparison matrix is such that the procedure described in I can be applied and the ratio with the original one can be calculated.

Denote the matrix on the right-hand side of (2.1) as A and the comparison matrix yet to be defined as $\bar{A}$. The guideline is to construct an $\bar{A}$ containing a submatrix $\bar{A}_0$ analogous to A_0, but with $\bar{C}(\phi)$ replacing $C(\phi)$ in (2.2). This should be done in such a way that we can evaluate $\det(A)/\det(\bar{A})$.

We can achieve the above by defining $\bar{A}$ as that obtained through adding a row and a column to A. This gives

$$\bar{A} = \begin{pmatrix} \bar{A}_0 + \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_0 + \bar{A}_{22} \end{pmatrix} \tag{2.4}$$

where, similar to those in A,

$$(\bar{A}_0)_{kj} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi\, e^{-i(k-j)\phi} \begin{pmatrix} 0 & \bar{C}(\phi) \\ -\bar{C}(\phi)^{-1} & 0 \end{pmatrix} \tag{2.5}$$

and

$$(\bar{A}_{pq})_{kj} = \sum_{s=1}^{2} (\bar{A}^s_{pq})_{kj} \tag{2.6a}$$

$$(\bar{A}^s_{pq})_{kj} = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi\, e^{-i(k-j)\phi} X^s_{pq}(\phi)\, \bar{b}^s_{pq}(\phi) \tag{2.6b}$$

with

$$\bar{b}^s_{pq} = \begin{pmatrix} \bar{C}_- \\ (-1)^{ps}(-i\bar{C}_+^{-1}) \end{pmatrix} [(-1)^{qs}(i\bar{C}_-^{-1}), \bar{C}_+] \tag{2.7}$$

The functions $\bar{C}_\pm$ come from the canonical factorization of $\bar{C}$; explicitly,

$$\bar{C}_+(\phi) = \bar{C}_-(-\phi)^{-1} = (1-\alpha_1 e^{i\phi})^{1/2}(1-\alpha_2^{-1}e^{i\phi})^{1/2} \tag{2.8}$$

where α_1 and α_2, defined in (I.A.2), are functions of $\tanh(E_j/kT)$, $j=1, 2$.

The above-T_c spin correlation will be calculated by multiplying and dividing (2.1) by $\det(\bar{A})$,

$$\langle \sigma_{l,0}\sigma_{m,n} \rangle = (\det \bar{A})^{1/2} (\det A/\det \bar{A})^{1/2} \tag{2.9}$$

The calculation for $\det(\bar{A})$ is completely analogous to that for $\det(A)$ in I, and the result is given in Appendix A. Here we concentrate on the ratio, which has no analog in I.

Since deleting the first row and column of $\bar{A}$ gives A, the ratio of their determinants is, by Jacobi's theorem,[8]

$$\langle \sigma_{l,0} \sigma_{m,n} \rangle = (\det \bar{A})^{1/2} \, |(\bar{A}^{-1})_*| \tag{2.10}$$

in which the subscript asterisk on the right-hand side represents the upper leftmost element in the upper right quadrant of the matrix.

We will compute $(\bar{A}^{-1})_*$ in the next section. The computation bears some similarity with that in I, because we again make an expansion around $\bar{A}_0$, using the fact that we know $\bar{A}_0^{-1}$. More precisely, what we need is its Fourier transform:

$$\sum_{k,j=0}^{\infty} (\bar{A}_0^{-1})_{kj} \, e^{i(k\phi - j\phi)} = \frac{\bar{a}^+(\phi)\, \bar{a}^-(\theta)}{1 - e^{i(\phi - \theta + i\varepsilon)}} \tag{2.11}$$

with $\varepsilon \to 0^+$ and

$$\bar{a}^+(\phi) = \begin{pmatrix} 0 & -\bar{C}_+(\phi) \\ \bar{C}_+(\phi)^{-1} & 0 \end{pmatrix}, \qquad \bar{a}^-(\phi) = \begin{pmatrix} \bar{C}_-(\phi)^{-1} & 0 \\ 0 & \bar{C}_-(\phi) \end{pmatrix} \tag{2.12}$$

Formulas (2.11) and (2.12) are completely analogous to the below-T_c case.

In addition, we need

$$\begin{aligned} &\sum_{k=0}^{\infty} (\bar{A}_0^{-1})_{k0} \, e^{ik\phi} = \bar{a}^+(\phi) \\ &\sum_{j=0}^{\infty} (\bar{A}_0^{-1})_{0j} \, e^{-ij\theta} = -i\sigma_y \bar{a}^-(\theta) \end{aligned} \tag{2.13}$$

where σ_y is the Pauli matrix. The need for (2.13) arises in the calculation of $(\bar{A}^{-1})_*$, because the expression is no longer cyclic as in the low-temperature case, but has ends at the $k = j = 0$ block.

3. DISPERSION EXPANSION

As outlined above, we calculate $(\bar{A}^{-1})_*$ by separating out the $\bar{A}_0$ part to make an expansion for $\bar{A}^{-1}$; we use the identity

$$\bar{A}^{-1} = (1 + \bar{R})^{-1} (\bar{A}_0^{-1} \otimes I_2) = \sum_{k=0}^{\infty} (-1)^k \, \bar{R}^k (\bar{A}_0^{-1} \otimes I_2) \tag{3.1}$$

where I_2 is the two by two unit matrix, and

$$\bar{R} = \begin{pmatrix} \bar{A}_0^{-1} \bar{A}_{11} & \bar{A}_0^{-1} \bar{A}_{12} \\ \bar{A}_0^{-1} \bar{A}_{21} & \bar{A}_0^{-1} \bar{A}_{22} \end{pmatrix} \tag{3.2}$$

To get the desired element of (3.1), we first compute $\bar{R}^k$ in terms of $\bar{A}_0$ and $\bar{A}_{pq}$. It is straightforward to show that

$$[\bar{R}^k(\bar{A}_0^{-1}\times I_2)]_*$$
$$= \sum_{\{p_j, s_j\}} \sum_{\{n_i\}} \prod_{j=1}^{k} [(\bar{A}_0^{-1})_{n_{2j-2}, n_{2j-1}}(\bar{A}^{s_j}_{p_{j-1}p_j})_{n_{2j-1}, n_{2j}}](\bar{A}_0^{-1})_{n_{2k}, 0} \qquad (3.3)$$

where the summations are over $p_j = 1, 2$, $j = 1, 2,..., k-1$, with $p_0 = 1$, $p_k = 2$; $s_j = 1, 2$, $j = 1, 2,..., k$; and over $n_i = 0,..., \infty$, $i = 1,..., 2k$, with $n_0 = 0$.

Substituting (2.6) in (3.3), and using Eqs. (2.11) and (2.13), we are led to multiple integrals containing expressions of the form

$$(1, 0)(-i\sigma_y) \prod_{j=1}^{k} [\bar{a}^-(\phi_j)\, \bar{b}^{s_j}_{p_{j-1}p_j}(\phi_j)\, \bar{a}^+(\phi_j)] \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (3.4)$$

The explicit formulas for the 2×2 matrices $\bar{a}^{\pm}(\phi)$ and $\bar{b}^s_{pq}(\phi)$ are given in (2.12) and (2.7), respectively; and (3.4) can be easily computed. With the above, the dispersion expansion is obtained.

Thus the desired expansion is the following:

$$\langle \sigma_{l,0}\sigma_{m,n}\rangle = (\det \bar{A})^{1/2} \left| \sum_{k=1}^{\infty} \left(\frac{-i}{2\pi}\right)^k \sum_{\{p_j, s_j\}} \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_k \right.$$
$$\left. \times\ \bar{H}^{s_1}_{p_1}(\phi_1)\, \bar{K}^{s_1 s_2}_{p_1 p_2}(\phi_1, \phi_2)\, \bar{K}^{s_2 s_3}_{p_2 p_3}(\phi_2, \phi_3) \cdots \bar{K}^{s_{k-1} s_k}_{p_{k-1} 2}(\phi_{k-1}, \phi_k) \right|$$
$$(3.5a)$$

where the summations over p_j and s_j are those following (3.3); the integrand is reduced to $\bar{H}^s_2(\phi_1)$ if $k = 1$; and the kernel is given by

$$\bar{K}^{st}_{pq}(\phi, \theta) = \frac{X^t_{pq}(\theta)}{1 - e^{i(\phi - \theta + i\varepsilon)}} \left[1 - (-1)^{p(s+t)} \frac{\bar{C}_+}{\bar{C}_-}(\phi) \frac{\bar{C}_-}{\bar{C}_+}(\theta) \right] \qquad (3.5b)$$

and

$$\bar{H}^s_p(\phi) = i(-1)^s X^s_{1p}(\phi)\, \bar{C}_-(\phi)\, \bar{C}_+(\phi)^{-1} \qquad (3.5c)$$

4. CONTINUUM LIMIT AND ASYMPTOTIC BEHAVIORS

The critical temperature T_c for the defect system is the same as for its homogeneous special case, and is given by (1.2). We consider the scaling limit, which is the limit for $T \to T_c$, the correlation length $\xi \to \infty$, and the scaled distance $r \sim R/\xi$ is kept fixed.

Let ξ_h and ξ_v be the horizontal and vertical correlation lengths, respectively, in the pure system. Near T_c, they diverge linearly with T as in

(I.4.7). Define the scaled distance, x, the y's, and the scaled integration variable w as follows:

$$x=\frac{n}{\xi_h}, \qquad y_1=\frac{|l|}{\xi_v}, \qquad y_2=\frac{m}{\xi_v}, \qquad y=\frac{m-l}{\xi_v}, \qquad \bar{y}=\frac{m+l}{\xi_v}, \qquad \phi=\frac{w}{\xi_h} \tag{4.1}$$

It can be shown that the scaling form of (3.5) is

$$\lim\langle\sigma_{l,0}\sigma_{m,n}\rangle(1-\alpha_1)(\det \bar{A})^{-1/2}$$
$$=\left|\sum_{k=1}^{\infty}\left(\frac{-i}{2\pi}\right)^k \sum_{\{p_j,s_j\}}\int_{-\infty}^{\infty}dw_1\cdots\int_{-\infty}^{\infty}dw_k \right.$$
$$\left.\times \bar{F}_{p_1}^{s_1}(w_1)\,\bar{L}_{p_1p_2}^{s_1s_2}(w_1,w_2)\,\bar{L}_{p_2p_3}^{s_2s_3}(w_2,w_3)\cdots\bar{L}_{p_{k-1}2}^{s_{k-1}s_k}(w_{k-1},w_k)\right| \tag{4.2a}$$

$$\bar{L}_{pq}^{st}(w,\rho)=\frac{Y_{pq}^{t}(\rho)}{(1+\rho^2)^{1/2}}\,\frac{(1+\rho^2)^{1/2}-(-1)^{p(s+t)}(1+w^2)^{1/2}}{w-\rho+i\varepsilon} \tag{4.2b}$$

$$\bar{F}_p^s(w)=-(-1)^s\,Y_{1p}^s(w)(1+w^2)^{-1/2} \tag{4.2c}$$

where the function $Y_{pq}^s(w)$, coming from scaling $X_{pq}^s(\phi)$, is given in (I.A.12), and the constant α_1 is given in (I.A.2). Since this reduces to that of the pure system[9] and the convergence problem there has been investigated,[10] our results should be valid for a range of parameters.

Next we find the asymptotic behavior for (4.2) in the limit y_1, y_2, and $x\to\infty$. In the pure case this is

$$\langle\sigma_0\sigma_r\rangle=e^{-r}/(2\pi r)^{1/2} \tag{4.3}$$

Note that we have absorbed a renormalization factor $\bar{M}^2$, the disorder expectation, in the left-hand side. For the defect model, the correlation function below is renormalized by the local value $\bar{M}_l\bar{M}_m$.

Equation (4.3) is of the Ornstein–Zernike form,[11] since there is no special symmetry to destroy the OZ behavior as in the low-temperature case. Although there is this basic difference between the low- and high-temperature phases, the main features peculiar to the defect system are quite similar. These include the very different behaviors for the $l<0<m$ and $0<l<m$ cases and the appearance of defect-dependent correlation lengths.

The defect strength in the scaling limit is defined as

$$\tau=\operatorname{sgn}(T-T_c)\tanh[2(2E_1'-E_1-E_2'^*)/kT_c] \tag{4.4}$$

where the asterisk denotes the dual in the sense of (1.1).

Let

$$\langle \sigma_{y_1,0}\sigma_{y_2,x}\rangle_\tau = f_\tau(r,\bar{r},y)\langle \sigma_0\sigma_r\rangle_0 + \tau g_\tau(r,\bar{r},y) \tag{4.5}$$

In the pure limit, $\tau=0$ and $f_0(\cdots)=1$; in general, f_τ and g_τ are the following:

1. for $y_1 \ll 0 \ll y_2$,

$$f_\tau(r,\bar{r},y) = \bar{\tau}y/(y-r\tau) \tag{4.6}$$

$$g_\tau(r,\bar{r},y) = \Theta(\tau - y/r)\exp(-y\tau - x\bar{\tau}) \tag{4.7}$$

2. for $0 \ll y_1 \ll y_2$,

$$f_\tau(r,\bar{r},y) = 1 \tag{4.8}$$

$$g_\tau(r,\bar{r},y) = \frac{x}{\bar{y}-\bar{r}\tau}\frac{\exp(-\bar{r})}{(2\pi\bar{r})^{1/2}} - (1-\tau)^{-1}\frac{r}{r+y}\frac{\exp(-r-2y_1)}{2\pi(2y_1)^{3/2}r^{1/2}} + \Theta\left(\tau - \frac{\bar{y}}{\bar{r}}\right)\exp(-\bar{y}\tau - x\bar{\tau}) \tag{4.9}$$

where $\Theta(x)$ is the step function, and

$$r^2 = x^2 + y^2, \qquad \bar{r}^2 = x^2 + \bar{y}^2, \qquad \bar{\tau}^2 = 1 - \tau^2 \tag{4.10}$$

The above result is obtained by a steepest descent analysis on the leading integrals in the expansion (4.2); all the other terms are exponentially smaller. The factor $\det(\bar{A})$, being analogous to the low-temperature case, has the asymptotic behavior

$$(\det \bar{A})^{1/2} \approx \bar{M}_l \bar{M}_m[1 + O(e^{-2r})] \tag{4.11}$$

Hence, it does not contribute to the leading exponential decay. The steepest descent analysis is similar to that shown in Appendix B of I, and is omitted here.

The terms containing defect-dependent correlation lengths (4.7) and (4.9) dominate when the Θ function is one. The conditions for this to occur is the same as in the low-temperature case, and is shown in Fig. 3 in paper I.

There is also the situation where one spin is on the defects. Although this case is contained in the dispersion expansions derived here and in I if the proper limit is taken, it has not been considered asymptotically because it is necessary to require both y_1 and y_2 large, in order to reduce the expansions to a finite number of terms. In Appendix C we derive dispersion expansions for this special case in the line-defect model for both $T < T_c$ and $T > T_c$. Those expansions are suitable for asymptotic analysis. It is reasonable to expect that by replacing the defect strength there by the general τ of (4.4) in the scaling limit, the result should be valid for the more general linear-defect model considered in this paper.

The study of correlation functions at criticality has drawn much attention due to its connection with conformal algebra.[12] The dispersion expansions diverge at criticality, as the scaled distances approach zero. However, it is known that for the pure[9,13] and the half-plane[14] models, the dispersion expansions can be shown to satisfy nonlinear differential equations; the critical correlations are then obtainable from studying the nonlinear equations. A preliminary attempt to find if such equations exist for the defect model, following the method of ref. 13, excludes the possibility of a simple extension from the pure model.

APPENDIX A. EXPANSIONS FOR DET($\bar{A}$)

Here we give the expansions for $(\det \bar{A})^{1/2}$ on the lattice and in the scaling limit. The matrix $\bar{A}$, defined in (2.4)–(2.7), is identical with matrix A if we let $C(\phi) \to \bar{C}(\phi)$ in A. We can therefore write down the expansions, using the below-T_c result, without much effort.

On the lattice,

$$(\det \bar{A})^{1/2} = \bar{M}^2 \exp\left[-\sum_{k=1}^{\infty} \frac{1}{2k}\left(\frac{-1}{2\pi}\right)^k \sum_{\{p_j, s_j\}} \int_{-\pi}^{\pi} d\phi_1 \cdots \int_{-\pi}^{\pi} d\phi_k \right.$$
$$\left. \times\, \bar{K}^{s_1 s_2}_{p_1 p_2}(\phi_1, \phi_2)\, \bar{K}^{s_2 s_3}_{p_2 p_3}(\phi_2, \phi_3) \cdots \bar{K}^{s_k s_1}_{p_k p_1}(\phi_k, \phi_1) \right] \tag{A.1}$$

In the scaling limit,

$$(\det \bar{A})^{1/2} = \bar{M}_l \bar{M}_m \exp\left[-\sum_{k=2}^{\infty} \frac{1}{2k}\left(\frac{-i}{2\pi}\right)^k {\sum_{\{p_j, s_j\}}}' \int_{-\infty}^{\infty} dw_1 \cdots \int_{-\infty}^{\infty} dw_k \right.$$
$$\left. \times\, \bar{L}^{s_1 s_2}_{p_1 p_2}(w_1, w_2)\, \bar{L}^{s_2 s_3}_{p_2 p_3}(w_2, w_3) \cdots \bar{L}^{s_k s_1}_{p_k p_1}(w_k, w_1) \right] \tag{A.2}$$

The summations are over $p_j = 1, 2$, $s_j = 1, 2$, $j = 1, 2,..., k$, except that the terms with $p_1 = p_2 = \cdots = p_k$ are excluded in (A.2); and $\bar{K}^{st}_{pq}$ is given in (3.5b) and $\bar{L}^{st}_{pq}$ in (4.2b).

APPENDIX B. RELATION WITH GENERATING FUNCTIONS

In this paper and in paper I, we start by presenting the spin–spin correlation function as the determinant of the infinite matrix given in (2.1)–(2.3). Those formulas come from a more general expression valid for layered Ising models, namely

$$\langle \sigma_{l,0}\sigma_{m,n}\rangle = \det{}^{1/2}\begin{pmatrix} G(l,l)_{k_1 j_1} & G(l,m)_{k_1 j_2} \\ -G^T(l,m)_{k_2 j_1} & G(m,m)_{k_2 j_2}\end{pmatrix} \tag{B.1}$$

where $k_1, j_1 = -\infty,..., -1, 0$ and $k_2, j_2 = -\infty,..., 2n-1, 2n$. The generating function $G(l, m)_{kj}$ is defined in terms of the row-to-row transfer matrices T_i and the Clifford operators γ_i:

$$G(l,m)_{kj} = \frac{2i}{\operatorname{tr} T}\operatorname{tr}(\cdots T_{2l-1}T_{2l}^{1/2}\gamma_k T_{2l}^{1/2}T_{2l+1}\cdots T_{2m-1}T_{2m}^{1/2}\gamma_j T_{2m}^{1/2}T_{2m+1}\cdots) \tag{B.2}$$

where T is the ordered product of all transfer matrices.

The generating function for the linear-defect model has been computed in ref. 6. It was shown to have a 2×2 block Toeplitz form. Let $[k, j]$ denote the block

$$\begin{matrix} (2k+1, 2j+1) & (2k+1, 2j+2) \\ (2k+2, 2j+1) & (2k+2, 2j+2) \end{matrix} \tag{B.3}$$

The generating function has the form

$$G(l,m)_{[k,j]} = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, e^{-i(k-j)\phi}\hat{G}(l,m) \tag{B.4}$$

with $\hat{G}(l, m)$ a 2×2 matrix independent of k and j.

The formulas (2.1)–(2.3) arise from (B.1) by a substitution of the explicit expressions for the generating functions from ref. 6, and by a rearrangement of column and row indices. Specifically, the relation between the generating functions and the matrices in (2.1)–(2.3) is given by

$$\begin{aligned} (A_0 + A_{11})_{kj} &= G(l,l)_{[-k-1,-j-1]} \\ (A_0 + A_{22})_{kj} &= G(m,m)_{[-k-1,-j-1]} \\ (A_{12})_{kj} &= G(l,m)_{[-k-1,-j+n-1]} \\ (A_{21})_{kj} &= -[(A_{12})_{jk}]^T \end{aligned} \tag{B.5}$$

The ranges for the indices k and j are thus changed from those in (B.1) to the more convenient $[0, \infty]$ uniformly.

APPENDIX C. A SPECIAL CASE

The asymptotic behaviors computed in Section 4 for $T > T_c$ and in Section 5 of I for $T < T_c$ clearly do not include the important case in which one spin is on the defect line while the other is far away. The case where both spins are on the defect line has been computed[5] for the line-defect model. The line-defect model is the special case where $E_2' = \infty$ and the defect rows collapse into one line with coupling $2E_1'$. This model is particularly symmetric for the one-spin-on-defect case, and offers the possibility to derive the dispersion expansions in a unique form suitable for asymptotic analysis. Below we show the expansion for $T < T_c$ and then for $T > T_c$.

The guideline in making the expansion is to separate out the part of the matrix that can be inverted. This invertible matrix is A_0 for $T < T_c$ and $\bar{A}_0$ for $T > T_c$. In the special case where $E_2' = \infty$ and $l = 0$, it turns out that $A_0 + A_{11}$ of (2.1) is invertible for $T < T_c$.

Restricted to the special case, let

$$\langle \sigma_{0,0} \sigma_{m,n} \rangle = \det{}^{1/2} \begin{pmatrix} A_1 + A_{11} & A_{12} \\ A_{21} & A_2 + A_{22} \end{pmatrix} \tag{C.1}$$

Before giving the submatrices explicitly, we explain their connection with those in Appendix B. The submatrix A_2 is the same as A_0 in (2.2); A_{12}, A_{21}, and A_{22} are the same as in (2.1) restricted to this special case; A_{11} in (C.1) is zero, because we have grouped the whole block into A_1 and have kept the null matrix only for notational convenience; finally, the upper left quadrant, when evaluated in this special case, can be expressed in the form of (2.2) with $C(\phi)$ replaced by $K(\phi)$.

The function[5] $K(\phi)$ will be given below; here we emphasize that, like $C(\phi)$, it satisfies the condition that $\ln K(\phi)$ is continuous and periodic and hence its determinant and inverse are calculable; therefore it has a canonical factorization $K(\phi) = K_+(\phi) K_-(\phi)$. This function should not be confused with the kernel $K_{pq}^{st}(\phi, \theta)$ in the dispersion expansion.

The submatrices in (C.1) are of block Toeplitz form,

$$\begin{aligned} (A_p)_{kj} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, e^{-i(k-j)\phi} a_p(\phi) \\ (A_{pq})_{kj} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\phi \, e^{-i(k-j)\phi} X_{pq}(\phi) \, b_{pq}(\phi) \end{aligned} \tag{C.2}$$

The function X_{pq} and matrices a_p and b_{pq} are given by

$$\begin{aligned} &X_{11} = 0, \qquad X_{22} = \kappa(1 - C^2)(\kappa + C)^{-1} (1 + kC)^{-1} \exp(-2m\Gamma) \\ &X_{pq} = (1 - \kappa^2)^{1/2} S_{p+} S_{q-} (\kappa + C)^{-1} \exp[-m\Gamma + i(p-q) n\phi], \qquad p \neq q \end{aligned} \tag{C.3}$$

where the functions $C(\phi)$ and $\Gamma(\phi)$ are defined in Appendix A of I; and

$$a_1(\phi)=\begin{pmatrix} 0 & K(\phi) \\ -K(\phi)^{-1} & 0 \end{pmatrix}, \qquad a_2(\phi)=\begin{pmatrix} 0 & C(\phi) \\ -C(\phi)^{-1} & 0 \end{pmatrix}$$
$$b_{pq}=\begin{pmatrix} S_{p-} \\ (-1)^p\,(-iS_{p+}^{-1}) \end{pmatrix} ((-1)^q\,(iS_{q-}^{-1}),\ S_{q+}) \tag{C.4}$$

where $S_{p\pm}$ is the canonical factorization of S_p, and

$$S_1(\phi)=K(\phi), \qquad S_2(\phi)=C(\phi)$$
$$K(\phi)=K^{-1}(-\phi)=[\kappa+C(\phi)][1+\kappa C(\phi)]^{-1} \tag{C.5}$$
$$\kappa=\tanh[(2E_1'-E_1)/kT]$$

The rest of the derivation follows Sections 2 and 3 of I closely, with the result

$$\begin{aligned}\langle\sigma_{l,0}\sigma_{m,n}\rangle_{T<T_c} &= M_0 M \exp\Bigg[-\sum_{k=1}^{\infty}\frac{1}{2k}\left(\frac{-1}{2\pi}\right)^k \sum_{\{p_j\}}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_k \\ &\quad\times K_{p_1p_2}(\phi_1,\phi_2)\,K_{p_2p_3}(\phi_2,\phi_3)\cdots K_{p_kp_1}(\phi_k,\phi_1)\Bigg]\end{aligned} \tag{C.6}$$
$$K_{pq}(\phi,\theta)=\frac{X_{pq}(\theta)}{1-e^{i(\phi-\theta)}}\left[1-\frac{S_{p+}}{S_{p-}}(\phi)\frac{S_{p-}}{S_{p+}}(\theta)\right]$$

Similarly, the above T_c result is derived in the same way as in Sections 2 and 3. And

$$\begin{aligned}\langle\sigma_{l,0}\sigma_{m,n}\rangle_{T>T_c} &= (\det\bar{A})^{1/2}\Bigg|\sum_{k=1}^{\infty}\left(\frac{-1}{2\pi}\right)^k \sum_{\{p_j\}}\int_{-\pi}^{\pi}d\phi_1\cdots\int_{-\pi}^{\pi}d\phi_k \\ &\quad\times \bar{H}_{p_1}(\phi_1)\,\bar{K}_{p_1p_2}(\phi_1,\phi_2)\,\bar{K}_{p_2p_3}(\phi_2,\phi_3)\cdots\bar{K}_{p_{k-1}2}(\phi_{k-1},\phi_k)\Bigg|\end{aligned} \tag{C.7}$$
$$\bar{K}_{pq}(\phi,\theta)=\frac{\bar{X}_{pq}(\theta)}{1-e^{i(\phi-\theta)}}\left[1-\frac{\bar{S}_{p+}}{\bar{S}_{p-}}(\phi)\frac{\bar{S}_{p-}}{\bar{S}_{p+}}(\theta)\right]$$
$$\bar{H}_p(\phi)=-i\bar{X}_{1p}(\phi)\,\bar{K}_-(\phi)\,\bar{K}_+(\phi)^{-1}$$

where $\bar{X}_{pq}=X_{pq}\exp(-i\,|p-q|\,\phi)$.

REFERENCES

1. L. Onsager, *Phys. Rev.* **65**:117 (1944).
2. H. A. Kramers and G. H. Wannier, *Phys. Rev.* **60**:252 (1941).
3. L. F. Ko, *J. Stat. Phys.*, this issue.
4. D. B. Abraham, *J. Phys. A* **14**:L369 (1981).
5. M. E. Fisher, *Physica* (Utrecht) *A* **106**:28 (1981); L. P. Kadanoff and F. J. Wegner, *Phys. Rev. B* **4**:3989 (1971); A. C. Brown, *Phys. Rev. B* **25**:331 (1982); L. P. Kadanoff, *Phys. Rev. B* **24**:5382 (1981); M. E. Fisher and A. E. Ferdinand, *Phys. Rev. Lett.* **19**:169 (1967); B. M. McCoy and J. H. H. Perk, *Phys. Rev. Lett.* **44**:840 (1980); in *The Riemann Problem, Complete Integrability and Arithmetic Applications*, D. Chudnovsky and G. Chudnovsky, eds., *Lecture Notes in Mathematics*, Vol. 925 (Springer, Berlin, 1982), p. 12; L. F. Ko, H. Au-Yang, and J. H. H. Perk, *Phys. Rev. Lett.* **54**:1091 (1985).
6. L. F. Ko, Ph.D. thesis, State University of New York, Stony Brook (1986).
7. B. Kaufman and L. Onsager, *Phys. Rev.* **76**:1244 (1949).
8. B. M. McCoy and T. T. Wu, *The Two-Dimensional Ising Model* (Harvard University Press, Cambridge, Massachusetts, 1973).
9. T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, *Phys. Rev. B* **13**:316 (1976).
10. J. Palmer and C. Tracy, *Adv. Appl. Math.* **2**:329 (1981).
11. W. J. Camp and M. E. Fisher, *Phys. Rev. B* **6**:946 (1972); W. J. Camp, *Phys. Rev. B* **7**:3187 (1973).
12. A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *J. Stat. Phys.* **34**:763 (1984); *Nucl. Phys. B* **241**:333 (1984); J. L. Cardy, *Nucl. Phys. B* **240**[FS]:514 (1984).
13. B. M. McCoy, C. A. Tracy, and T. T. Wu, *J. Math. Phys.* **18**:1058 (1977).
14. R. Z. Bariev, *Sov. Phys. JEPT* **50**:613 (1979); *Theor. Math. Phys.* **40**:623 (1979); **42**:173 (1980).